

Intertwining operators for non self-adjoint Hamiltonians and bicoherent states¹

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Abstract

This paper is devoted to the construction of what we will call *exactly solvable models*, i.e. of quantum mechanical systems described by an Hamiltonian H whose eigenvalues and eigenvectors can be explicitly constructed out of some *minimal ingredients*. In particular, motivated by PT-quantum mechanics, we will not insist on any self-adjointness feature of the Hamiltonians considered in our construction. We also introduce the so-called bicoherent states, we analyze some of their properties and we show how they can be used for quantizing a system. Some examples, both in finite and in infinite-dimensional Hilbert spaces, are discussed.

¹This paper is dedicated to the memory of Syed Twareque Ali, much more than just a colleague!

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1 Introduction

In recent years a growing interest on non self-adjoint operators with real eigenvalues has spread within the communities of physicists and of mathematicians, both for their possible applications to concrete (e.g., gain and loss) systems, and for the peculiar properties of these operators, which turn out to produce rather interesting mathematics in the Hilbert space where they are defined. A very recent book of collected papers (mainly) on the physical aspects of similar operators is [1], while many mathematical peculiarities are discussed in [2]. Concerning the latter, the role of biorthogonal sets, the spectral properties of these operators and the existence of similarity maps and of equivalent, or not equivalent, scalar products are just some of the points considered by several researchers in recent years.

Another very hot topic in physics has to do with the so-called intertwining operators, [3]. The reason for this is that, during the years, these operators have been used to construct more and more exactly solvable quantum mechanical models, i.e. Hamiltonians (usually, self-adjoint) for which the eigenvalues and the eigenvectors can be deduced in a reasonably simple way, out of a given *seed Hamiltonian* H_1 , and of some suitable operator X which, in many concrete situations, is not invertible. X is an intertwining operator for H_1 and H_2 if $H_2X = XH_1$. Hence, if φ_n satisfies the eigenvalue equation $H_1\varphi_n = \epsilon_n\varphi_n$, then (at least if $\varphi_n \notin \ker(X)$) calling $\Psi_n = X\varphi_n$, this is an eigenstate of H_2 : $H_2\Psi_n = \epsilon_n\Psi_n$. Of course, when X is invertible, H_2 can be written as $H_2 = XH_1X^{-1}$. In this case H_1 and H_2 are usually called similar[26]. We will see examples of both these cases (X^{-1} exists or not) in Section 2.

In this paper, we merge these two arguments, proposing a somehow general setting in which both these topics can be discussed and used to construct more models which are *under control*, i.e., are described by certain Hamiltonian-like operators for which we can find eigenvalues and eigenvectors, even if we give up the assumption of self-adjointness of the Hamiltonian itself. Our analysis somehow extends what was first considered in [5], where this kind of problems was first analyzed. Also, we discuss how these results can be used to construct some extended version of coherent states, and which are their properties. In particular, we focus our attention on the existence itself of these vectors, since this is not granted, in general. Then, we will deduce the related resolutions of the identity, and we will discuss their nature of eigenstates of some particular lowering operators. Also, we will show their role in setting up a simple and interesting quantizing recipe.

This article is organized as follows:

In the next section we introduce the general framework for our intertwining operators for

non self-adjoint operators, and we discuss the details of this construction, considering different situations depending on the different properties of the particular intertwining operator considered. Section 3 is devoted to some examples of this framework. In Section 4 we introduce our bicoherent states, and we show how these can be explicitly constructed by discussing a concrete example. The conclusions are given in Section 5.

2 A general settings for non self-adjoint operators

Let us consider an operator Θ_1 on a Hilbert space \mathcal{H} , in general different from its adjoint Θ_1^\dagger , $\Theta_1 \neq \Theta_1^\dagger$, with eigenvalues ϵ_n and eigenvectors $\varphi_n^{(1)}$:

$$\Theta_1 \varphi_n^{(1)} = \epsilon_n \varphi_n^{(1)}, \quad (2.1)$$

for all $n \geq 0$. In particular, in this section we will assume that all the eigenvalues have multiplicity one, but not that they are all real. This is relevant, for instance, in connection with PT-quantum mechanics in the so-called broken phase, [6]. Quite often, in the literature, the assumption on the multiplicity of the eigenvalues is used just to simplify the notation, see [7] for instance. Here, on the other hand, this is a more serious requirement, often satisfied in concrete models (see [8]-[12] among others), since it produces important consequences. This will be made clear later on.

For simplicity's sake, we will consider quite often the case in which $\dim(\mathcal{H}) < \infty$. This is because, in this case, we don't need to worry about the domains of the operators since they are all necessarily bounded. Nevertheless, quite often all along the paper we will also discuss what happens for infinite-dimensional Hilbert spaces, since this case is relevant in several physical applications. Roughly speaking, what we essentially want to discuss here is the possibility of constructing, out of Θ_1 and of some other ingredient (see below), another (again, not necessarily self-adjoint) operator Θ_2 , whose eigenvalues and eigenstates can be deduced out of ϵ_n and $\varphi_n^{(1)}$. Moreover, since $\Theta_1 \neq \Theta_1^\dagger$ and $\Theta_2 \neq \Theta_2^\dagger$ in general, we are also interested in deducing, if this is possible, something on the eigenfamilies of Θ_1^\dagger and Θ_2^\dagger . To be more explicit, the operator Θ_2 we are looking for will be searched using some suitable (generalized) intertwining operator X , acting on \mathcal{H} , which could be invertible or not, bounded or not, and obeying (or not) further useful conditions, which will be clarified all along this section. In particular, in the rest of this section we will consider separately three different situations: we begin by assuming that X^{-1} exists. Then we see what happens if X^{-1} exists **and** XX^\dagger commutes with Θ_1 . Finally, we discuss what can be deduced when X admits no inverse at all, but we still have $[XX^\dagger, \Theta_1] = 0$.

In this case, see Section 2.3 below, we will need another assumption, i.e. the fact that $X^\dagger X$ has an inverse [27].

2.1 Possibility number 1: X^{-1} exists

This is the easiest, and the most common situation, widely studied in the literature. In fact, in this case, it is clear that the operator Θ_2 we are looking for can be naturally defined as

$$\Theta_2 = X^{-1}\Theta_1 X, \quad (2.2)$$

while its eigenvectors are $\tilde{\varphi}_n^{(2)} = X^{-1}\varphi_n^{(1)}$, for all n . The operator Θ_2 is surely well defined on all of \mathcal{H} if $\dim(\mathcal{H}) < \infty$. Otherwise, this is not granted, and we have to impose conditions on the domains. In this case, in fact, in order to have a well defined operator Θ_2 , the set of the vectors $f \in D(X)$, the domain of X , such that $Xf \in D(\Theta_1)$ and $\Theta_1 Xf \in D(X^{-1})$, must be dense in \mathcal{H} . The set of all these vectors is what we call $D(\Theta_2)$, the domain of Θ_2 .

Now, it is evident that $\Theta_2 \tilde{\varphi}_n^{(2)} = \epsilon_n \tilde{\varphi}_n^{(2)}$, for all n , and that $X\Theta_2 = \Theta_1 X$. This case is not particularly interesting for us, since it only means that Θ_1 and Θ_2 are similar (i.e. related, as in (2.2) by an invertible operator), and will not be considered further here. We just want to add that, if $\mathcal{F}_\varphi^{(1)} = \{\varphi_n^{(1)}\}$ is a basis for \mathcal{H} , then it exists an unique biorthogonal basis $\mathcal{F}_\psi^{(1)} = \{\psi_n^{(1)}\}$, [14], $\langle \varphi_n^{(1)}, \psi_k^{(1)} \rangle = \delta_{n,k}$, and the vectors $\psi_n^{(1)}$ turn out to be eigenvectors of Θ_1^\dagger , with eigenvalue $\overline{\epsilon_n}$:

$$\Theta_1^\dagger \psi_n^{(1)} = \overline{\epsilon_n} \psi_n^{(1)},$$

for all n . As for the eigenstates of Θ_2^\dagger , these can also be easily found and turn out to be $\tilde{\psi}_n^{(2)} = X^\dagger \psi_n^{(1)}$. In fact, if $\psi_n^{(1)} \notin \ker(X^\dagger)$, a direct computation shows that $\Theta_2^\dagger \tilde{\psi}_n^{(2)} = \overline{\epsilon_n} \tilde{\psi}_n^{(2)}$. The role of $\ker(X^\dagger)$ is important here and will be stressed all along the paper, starting already from Section 2.2.

2.2 Possibility number 2: X^{-1} exists and $[XX^\dagger, \Theta_1] = 0$

In this case, the situation is a bit more interesting and richer than before. Let us call $N_1 = XX^\dagger$, and let us introduce Θ_2 as in (2.2), $\tilde{\varphi}_n^{(2)} = X^{-1}\varphi_n^{(1)}$, and the new vectors $\varphi_n^{(2)} = X^\dagger \varphi_n^{(1)}$. Hence, as we will show now, something interesting can still be deduced, at least if $\varphi_n^{(1)} \notin \ker(X^\dagger)$. In this case, in fact, it is clear that, again, $\Theta_2 \tilde{\varphi}_n^{(2)} = \epsilon_n \tilde{\varphi}_n^{(2)}$. Moreover,

$$\Theta_2 \varphi_n^{(2)} = (X^{-1}\Theta_1 X) X^\dagger \varphi_n^{(1)} = X^{-1} X X^\dagger \Theta_1 \varphi_n^{(1)} = \epsilon_n X^\dagger \varphi_n^{(1)} = \epsilon_n \varphi_n^{(2)},$$

where we have used the fact that $[N_1, \Theta_1] = 0$. This means that $\varphi_n^{(2)}$ are also eigenstates of Θ_2 , with the same eigenvalues as $\tilde{\varphi}_n^{(2)}$. But, recalling that the eigenvalues are assumed to have multiplicity one, this implies that $\tilde{\varphi}_n^{(2)}$ and $\varphi_n^{(2)}$ must be proportional:

$$\tilde{\varphi}_n^{(2)} = k_n \varphi_n^{(2)} \quad (2.3)$$

for some non zero k_n . Of course, this proportionality between $\tilde{\varphi}_n^{(2)}$ and $\varphi_n^{(2)}$ could be lost when the multiplicity of some ϵ_n is larger than one.

Remarks:– (1) It is useful to observe that $k_n = 0$ if, and only if, $\varphi_n^{(1)} \in \ker(X^\dagger)$.

(2) Going back to the condition $[N_1, \Theta_1] = 0$, at least if one of these operators is unbounded, then it is convenient to assume (as it often happens in concrete examples) that a dense subset \mathcal{D} of \mathcal{H} exists which is stable under the action of N_1 and Θ_1 . If this is the case, then $[N_1, \Theta_1] = 0$ must be understood in the following way: $N_1 \Theta_1 f = \Theta_1 N_1 f$, for all $f \in \mathcal{D}$. Several quantum mechanical systems having this feature are considered in [13], in the context of the so-called \mathcal{D} pseudo-bosons.

It is clear that Θ_1 and Θ_2 still obey the intertwining relation $X \Theta_2 = \Theta_1 X$. It is also clear that this equality should only hold on a dense subset of \mathcal{H} if $\dim(\mathcal{H}) = \infty$ and if some of the operators involved are unbounded. An induction argument shows that $X \Theta_2 = \Theta_1 X$ can be extended to higher powers of Θ_1 and Θ_2 : $X \Theta_2^n = \Theta_1^n X$, for all $n \geq 1$. Even more: if $f(x)$ is a function which admits a power series expansion, with infinite convergence radius, then $X f(\Theta_2) = f(\Theta_1) X$. Again, these two last intertwining equations should be considered on some suitable domain in case of unbounded operators. Otherwise, and in particular if $\dim(\mathcal{H}) < \infty$, they hold in all of \mathcal{H} .

Taking now the adjoint of $X \Theta_2 = \Theta_1 X$ we get another, equivalent, relation: $\Theta_2^\dagger X^\dagger = X^\dagger \Theta_1^\dagger$. However, under the conditions we are considering here ($[N_1, \Theta_1] = 0$), it is also possible to deduce that $\Theta_2 X^\dagger = X^\dagger \Theta_1$ as well. In fact we have:

$$\Theta_2 X^\dagger = X^{-1} \Theta_1 X X^\dagger = X^{-1} X X^\dagger \Theta_1 = X^\dagger \Theta_1,$$

which is what we had to check. Of course, the same arguments as above imply also that $X \Theta_2^\dagger = \Theta_1^\dagger X$, $\Theta_2^n X^\dagger = X^\dagger \Theta_1^n$, for all $n \geq 1$, and so on.

Let us now go back to the commutativity condition $[N_1, \Theta_1] = 0$, and let us recall that $\varphi_n^{(1)}$ is an eigenstate of Θ_1 . It is not a big surprise to find that $\varphi_n^{(1)}$ is also an eigenstate of N_1 . In fact, equation (2.3) can be rewritten as $X^{-1} \varphi_n^{(1)} = k_n X^\dagger \varphi_n^{(1)}$, for all n such that $\varphi_n^{(1)} \notin \ker(X^\dagger)$. Hence, left multiplying both sides of this equality for X , we get $\varphi_n^{(1)} = k_n N_1 \varphi_n^{(1)}$ or, written in

a more convenient way,

$$N_1 \varphi_n^{(1)} = k_n^{-1} \varphi_n^{(1)}. \quad (2.4)$$

We observe that this equation is well defined since, as we have seen, $\varphi_n^{(1)} \notin \ker(X^\dagger)$ if and only if $k_n \neq 0$. Incidentally, since N_1 is a positive operator, this implies that k_n is strictly positive. The above implication can be inverted and, in fact, equation (2.3) can be easily deduced from (2.4). Therefore, these two equations are equivalent, at least in absence of domain problems. Needless to say, the (possibly) most interesting situation here is when the eigenvalues k_n^{-1} of N_1 are degenerate. In fact, if this is not so, the various $\varphi_n^{(1)}$'s turn out to be also eigenstates of the (at least formally) self-adjoint operator N_1 , and this would automatically imply that they are mutually orthogonal. When this happens, the two sets $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\psi^{(1)} = \{\psi_n^{(1)}\}$ essentially coincide.

2.3 Possibility number 3: X^{-1} does not exist

This is probably the most interesting situation, since it is quite different from what it is usually discussed in the literature. In fact, in this case, equation (2.2) makes no sense, but we can still see that, under suitable conditions on X , an operator Θ_2 can still be defined, with the feature that its eigensystem can be easily deduced out of the one of Θ_1 . This will be again related to the existence of an intertwining relation between Θ_1 and Θ_2 , which can be deduced also in the present situation. What will be discussed here extends what was originally considered in [5]. The working assumptions are the following: the operator $N_1 = XX^\dagger$ commutes with Θ_1 as in Section 2.2, while the operator $N_2 = X^\dagger X$ is strictly positive, hence invertible. As usual, we will mainly work under the assumption that $\dim(\mathcal{H}) < \infty$, to avoid any domain problem. However, also in view of the examples given in Section 3 and of the application to coherent states in Section 4, we will sometimes comment on the infinite dimensional case.

Under our assumptions, we introduce now

$$\Theta_2 = N_2^{-1} (X^\dagger \Theta_1 X), \quad (2.5)$$

while $\varphi_n^{(2)}$ are defined as in Section 2.2, $\varphi_n^{(2)} = X^\dagger \varphi_n^{(1)}$. Once again, the interesting situation is when $\varphi_n^{(1)} \notin \ker(X^\dagger)$ to ensure that $\varphi_n^{(2)} \neq 0$. But, as we will see in some explicit examples, this is not always granted. When this happens, the set of the eigenvalues of Θ_2 turns out to be a proper subset of the set of eigenvalues of Θ_1 . We will meet explicitly with this situation in, e.g., Section 3.2.

A first obvious remark is that, in this case, equation (2.3) makes no sense, since the vector $\tilde{\varphi}_n^{(2)}$ cannot even be defined. However, using our assumptions, we can still check that $\Theta_2 \varphi_n^{(2)} =$

$\epsilon_n \varphi_n^{(2)}$. The proof, which can be already found in [5], goes like this

$$\Theta_2 \varphi_n^{(2)} = N_2^{-1} (X^\dagger \Theta_1 X) (X^\dagger \varphi_n^{(1)}) = N_2^{-1} X^\dagger N_1 \Theta_1 \varphi_n^{(1)} = N_2^{-1} N_2 X^\dagger (\epsilon_n \varphi_n^{(1)}) = \epsilon_n \varphi_n^{(2)},$$

and uses the fact that $\Theta_1 N_1 = N_1 \Theta_1$. Now it is also interesting to notice that, despite of the fact that definition (2.5) is significantly different from (2.2), Θ_2 and Θ_1 still satisfy the same intertwining relations as those found previously. To prove this, we first observe that $[N_2, X^\dagger \Theta_1 X] = 0$:

$$N_2 X^\dagger \Theta_1 X = X^\dagger X (X^\dagger \Theta_1 X) = X^\dagger N_1 \Theta_1 X = X^\dagger \Theta_1 N_1 X = X^\dagger \Theta_1 X X^\dagger X = X^\dagger \Theta_1 X N_2.$$

Then we also have $[N_2^{-1}, X^\dagger \Theta_1 X] = 0$. Of course, this is true with no further assumption if $\dim(\mathcal{H}) < \infty$, while some more care is needed otherwise. Now we have

$$X \Theta_2 = X (X^\dagger \Theta_1 X) N_2^{-1} = N_1 \Theta_1 X N_2^{-1} = \Theta_1 N_1 X N_2^{-1} = \Theta_1 X N_2 N_2^{-1} = \Theta_1 X.$$

As for the second intertwining relation, $X^\dagger \Theta_1 = \Theta_2 X^\dagger$, we have

$$\Theta_2 X^\dagger = N_2^{-1} (X^\dagger \Theta_1 X) X^\dagger = N_2^{-1} X^\dagger \Theta_1 N_1 = N_2^{-1} X^\dagger N_1 \Theta_1 = N_2^{-1} N_2 X^\dagger \Theta_1 = X^\dagger \Theta_1,$$

as we had to prove. Moreover, it is clear that the following third intertwining relation is also satisfied: $X N_2 = N_1 X$, which shows that X is also an intertwining operator between the operators N_1 and N_2 , and not only between Θ_1 and Θ_2 .

The following result replace, and correct, a similar statement in [5]:

Proposition 2.1 *If Θ_1 , N_1 and N_2 are as above, and in particular if $[\Theta_1, N_1] = 0$ and $N_2 > 0$, then: (i) $[N_2, \Theta_2] = 0$; (ii) if $\Theta_1 = \Theta_1^\dagger$, then $\Theta_2 = \Theta_2^\dagger$; (iii) if $\Theta_2 = \Theta_2^\dagger$ and if $N_1 > 0$, then $\Theta_1 = \Theta_1^\dagger$.*

Proof –

(i) This claim follows from the definition (2.5) of Θ_2 and from the equality $[N_2, X^\dagger \Theta_1 X] = 0$, which we have already proved.

(ii) Let $\Theta_1 = \Theta_1^\dagger$. Then, since $N_2^\dagger = N_2$,

$$\Theta_2^\dagger = (N_2^{-1} (X^\dagger \Theta_1 X))^\dagger = (X^\dagger \Theta_1 X) N_2^{-1} = \Theta_2.$$

(iii) Let now assume that $N_1 > 0$, so that N_1^{-1} exists, and that $\Theta_2 = \Theta_2^\dagger$. Then, left-multiplying this last equality by X and right-multiplying it by X^\dagger , we get $X \Theta_2 X^\dagger = X \Theta_2^\dagger X^\dagger$. Now, since $X \Theta_2 = \Theta_1 X$ and $X \Theta_2^\dagger = \Theta_1^\dagger X$, we get $X \Theta_2 X^\dagger = \Theta_1 X X^\dagger = \Theta_1 N_1$ and $X \Theta_2^\dagger X^\dagger = \Theta_1^\dagger X X^\dagger = \Theta_1^\dagger N_1$. Hence $\Theta_1 N_1 = \Theta_1^\dagger N_1$ which, since N_1^{-1} exists, implies our statement. \square

Remarks:– (1) Similar results can be deduced also in the context of Section 2.2, but we will not repeat the proofs here.

(2) In [5] it was stated that $\Theta_1 = \Theta_1^\dagger$ if and only if $\Theta_2 = \Theta_2^\dagger$, with no requirement on the invertibility of N_1 . This was not correct. We notice that the statement here is more natural than the one in [5], since the role of N_1 and N_2 is now completely symmetric.

(3) Under the same assumptions of Proposition 2.1 it is possible to check that $XN_2^{-1} = N_1^{-1}X$ and that $\Theta_1 = N_1^{-1}(X\Theta_2X^\dagger)$. This formula is completely analogous to the one in (2.5).

2.4 The eigensystems for Θ_1^\dagger and Θ_2^\dagger

Starting now with the set $\mathcal{F}_\varphi^{(1)} = \{\varphi_n^{(1)}\}$ of eigenstates of Θ_1 , see (2.1), and assuming that this set is a basis for \mathcal{H} , it is possible to define uniquely, [14], a second set, $\mathcal{F}_\psi^{(1)} = \{\psi_n^{(1)}\}$, which is still a basis for \mathcal{H} and is biorthogonal to $\mathcal{F}_\varphi^{(1)}$: $\langle \varphi_k^{(1)}, \psi_n^{(1)} \rangle = \delta_{k,n}$. It is clear that both these sets are complete: if $f \in \mathcal{H}$ is orthogonal to all the $\varphi_n^{(1)}$'s (or to all the $\psi_n^{(1)}$'s), f is necessarily zero. Then it is easy to check that, not surprisingly, the vectors in $\mathcal{F}_\psi^{(1)}$ are eigenstates of Θ_1^\dagger :

$$\Theta_1^\dagger \psi_n^{(1)} = \overline{\epsilon_n} \psi_n^{(1)}, \quad (2.6)$$

for all n . In fact we have

$$\left\langle \left(\Theta_1^\dagger \psi_n^{(1)} - \overline{\epsilon_n} \psi_n^{(1)} \right), \varphi_n^{(1)} \right\rangle = \left\langle \psi_n^{(1)}, \Theta_1 \varphi_n^{(1)} \right\rangle - \epsilon_n \left\langle \psi_n^{(1)}, \varphi_n^{(1)} \right\rangle = 0$$

for all n and k . Formula (2.6) now follows from the completeness of $\mathcal{F}_\varphi^{(1)}$.

Now we want to construct, out of Θ_1^\dagger , a second operator, which we could call $(\Theta_1^\dagger)_2$, mimicking what we have done in (2.5). This is possible since $[N_1, \Theta_1] = 0$ implies that $[N_1, \Theta_1^\dagger] = 0$ as well. Hence we are under the working assumptions listed at the beginning of this section, with Θ_1 replaced by Θ_1^\dagger . Therefore we can define

$$(\Theta_1^\dagger)_2 = N_2^{-1} \left(X^\dagger \Theta_1^\dagger X \right). \quad (2.7)$$

It is interesting to notice that, if we take the adjoint of Θ_2 in (2.5) and we use the fact that $[N_2^{-1}, X^\dagger \Theta_1 X] = 0$, we get $(\Theta_1^\dagger)_2 = \Theta_2^\dagger$. In other words: taking the adjoint of the operator Θ_2 in (2.5) or defining, again as in (2.5), the new operator in (2.7), makes no difference.

Let us now go back to the existence of the non zero vector $\varphi_n^{(2)}$. As we have already discussed, $\varphi_n^{(2)} \neq 0$ if and only if $\varphi_n^{(1)} \notin \ker(X^\dagger)$. In [5] we have shown that $\varphi_n^{(1)} \in \ker(X^\dagger)$ if and only if $\varphi_n^{(2)} \in \ker(X)$. It is now easy to check that, if $\varphi_n^{(1)} \notin \ker(X^\dagger)$, then $\varphi_n^{(1)} \notin \ker(N_1)$ and $\varphi_n^{(2)} \notin \ker(N_2)$. In the examples and in Section 4 this aspect will be considered further.

Now, following what we did for $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\varphi^{(2)}$, we define new vectors $\psi_n^{(2)} = X^\dagger \psi_n^{(1)}$, and the related set $\mathcal{F}_\psi^{(2)} = \{\psi_n^{(2)}\}$. Of course, $\psi_n^{(2)} \neq 0$ if $\psi_n^{(1)} \notin \ker(X^\dagger)$.

A direct computation, based on the fact that $[N_1, \Theta_1^\dagger] = 0$, shows that each $\psi_n^{(2)} \neq 0$ is an eigenstate of Θ_2^\dagger :

$$\Theta_2^\dagger \psi_n^{(2)} = N_2^{-1} (X^\dagger \Theta_1^\dagger X) X^\dagger \psi_n^{(1)} = N_2^{-1} X^\dagger N_1 \Theta_1^\dagger \psi_n^{(1)} = X^\dagger \overline{\epsilon}_n \psi_n^{(1)} = \overline{\epsilon}_n \psi_n^{(2)}.$$

We will now prove that an eigenvalue equation like the one in (2.4), deduced under the assumption that X^{-1} does exist, can also be found now, and can even be extended to N_2 . What we only need is that the eigenvalues ϵ_n of Θ_1 have multiplicity one, and that $\varphi_n^{(1)} \notin \ker(X^\dagger)$. In fact, under this last condition, $N_1 \varphi_n^{(1)} \neq 0$. Moreover, since $\Theta_1(N_1 \varphi_n^{(1)}) = N_1(\Theta_1 \varphi_n^{(1)}) = \epsilon_n(N_1 \varphi_n^{(1)})$, the vector $N_1 \varphi_n^{(1)}$ must be proportional to $\varphi_n^{(1)}$ itself. To distinguish here the situation with respect to that considered in Section 2.2, we call \tilde{k}_n (instead of k_n^{-1}) this proportionality constant. Hence we get

$$N_1 \varphi_n^{(1)} = \tilde{k}_n \varphi_n^{(1)}. \quad (2.8)$$

Now, left-multiplying both sides of this equation with X^\dagger we get $(X^\dagger X)(X^\dagger \varphi_n^{(1)}) = \tilde{k}_n X^\dagger \varphi_n^{(1)}$, which can be rewritten as

$$N_2 \varphi_n^{(2)} = \tilde{k}_n \varphi_n^{(2)}. \quad (2.9)$$

Hence N_1 and N_2 have the same eigenvalue \tilde{k}_n , at least for those n 's for which $\varphi_n^{(1)} \notin \ker(X^\dagger)$. Notice also that, for all these n , we get

$$\tilde{k}_n = \left(\frac{\|\varphi_n^{(2)}\|}{\|\varphi_n^{(1)}\|} \right)^2 > 0.$$

As already observed previously, it is clear that, if the multiplicity of the eigenvalues \tilde{k}_n of N_1 and N_2 is one, this makes of $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\varphi^{(2)}$ two different sets each made of mutually orthogonal vectors. Hence, if we are interested to non-o.n. sets of eigenvectors of Θ_1 and Θ_2 , it is crucial that not all the \tilde{k}_n have multiplicity one.

Equation (2.8) can also be written as $X X^\dagger \varphi_n^{(1)} = \tilde{k}_n \varphi_n^{(1)}$, i.e. as $X \varphi_n^{(2)} = \tilde{k}_n \varphi_n^{(1)}$, which implies that $\varphi_n^{(1)} = \frac{1}{\tilde{k}_n} X \varphi_n^{(2)}$. This equality can be considered as the *inverse* of the equation $\varphi_n^{(2)} = X^\dagger \varphi_n^{(1)}$ in our context.

Summarizing the main outcome of our analysis we can say that *out of the triple $(\Theta_1, \{\epsilon_n\}, \mathcal{F}_\varphi^{(1)})$ we can produce three more triples which are associated, in the way discussed before, to other exactly solvable models: $(\Theta_2, \{\epsilon_n\}, \mathcal{F}_\varphi^{(2)})$, $(\Theta_1^\dagger, \{\bar{\epsilon}_n\}, \mathcal{F}_\Psi^{(1)})$ and $(\Theta_2^\dagger, \{\bar{\epsilon}_n\}, \mathcal{F}_\Psi^{(2)})$.*

It should be noticed that an important difference between the pairs $(\mathcal{F}_\varphi^{(1)}, \mathcal{F}_\psi^{(1)})$ and $(\mathcal{F}_\varphi^{(2)}, \mathcal{F}_\psi^{(2)})$ exists, and is the following: while the first pair satisfies the orthonormality condition $\langle \varphi_k^{(1)}, \psi_n^{(1)} \rangle = \delta_{k,n}$ by construction, the second satisfies the slightly different condition: $\langle \varphi_k^{(2)}, \psi_n^{(2)} \rangle = \tilde{k}_n \delta_{k,n}$. In fact, we have

$$\langle \varphi_k^{(2)}, \psi_n^{(2)} \rangle = \langle X^\dagger \varphi_k^{(1)}, X^\dagger \psi_n^{(1)} \rangle = \langle N_1 \varphi_k^{(1)}, \psi_n^{(1)} \rangle = \tilde{k}_n \langle \varphi_k^{(1)}, \psi_n^{(1)} \rangle = \tilde{k}_n \delta_{k,n}. \quad (2.10)$$

Of course, strict biorthonormality could be recovered by changing a little bit the definition of, say $\psi_n^{(2)}$, by replacing the original formula, $\psi_n^{(2)} = X^\dagger \psi_n^{(1)}$, with $\psi_n^{(2)} = \frac{1}{\tilde{k}_n} X^\dagger \psi_n^{(1)}$. However, this would break down the original symmetry between $(\mathcal{F}_\varphi^{(1)}, \mathcal{F}_\psi^{(1)})$ and $(\mathcal{F}_\varphi^{(2)}, \mathcal{F}_\psi^{(2)})$, and we prefer not to adopt this alternative definition here.

It is now interesting to stress that, in all the situations considered above, i.e. both when X^{-1} exists and when it doesn't, we have been able to deduce intertwining relations between Θ_1 and Θ_2 . In particular, independently of the particular definition adopted here for Θ_2 , (2.2) or (2.5), we have always deduced that $X\Theta_2 = \Theta_1 X$. The Proposition below shows that this equation is really a key feature of our formulation, and not just a consequence of some *smart* definitions.

Proposition 2.2 *Suppose two operators Θ_1 and Θ_2 satisfy the equation $X\Theta_2 = \Theta_1 X$, for some suitable X . Hence: (i) if X is invertible, then $\Theta_2 = X^{-1}\Theta_1 X$; (ii) if X is not invertible, but $N_2 = X^\dagger X$ is invertible, then $\Theta_2 = N_2^{-1}(X^\dagger \Theta_1 X)$. In this case, taken an eigenvector $\varphi_n^{(1)}$ of Θ_1 , if we further have $[X X^\dagger, \Theta_1] = 0$ and if $\varphi_n^{(1)} \notin \ker(X^\dagger)$, then $\varphi_n^{(2)} = X^\dagger \varphi_n^{(1)}$ is an eigenvector of Θ_2 , with the same eigenvalue as $\varphi_n^{(1)}$.*

Proof – The proof of (i) is trivial. As for (ii), we observe that left-multiplying $X\Theta_2 = \Theta_1 X$ for X^\dagger , and using the fact that $N_2 = X^\dagger X$ is invertible, we deduce that $\Theta_2 = N_2^{-1}(X^\dagger \Theta_1 X)$. The rest of the statement is clear. □

Remarks:– (1) In [7] we have discussed the role of antilinear operators for some intertwining relations between operators with complex eigenvalues. At a first sight, this might appear not extremely far away from what we have done here. However, the two situations are indeed very different since the assumptions on N_1 and N_2 considered here play a crucial role in what is done in this section, while similar conditions are completely absent in [7].

(2) It may be interesting to notice that the above general scheme can be easily extended to a slightly more general situation, i.e. to the case in which Θ_1 and Θ_2 act on two different Hilbert spaces, \mathcal{H}_1 and \mathcal{H}_2 . In this case, of course, X , X^{-1} (if it exists) and X^\dagger are operators between \mathcal{H}_1 and \mathcal{H}_2 or viceversa, N_1 and N_2 also act on different Hilbert spaces, and $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$ belong to these different spaces. However, with some minor (and obvious) modifications, all the results proved in this section can again be deduced. We will see a concrete example of this situation in Section 3.2.

3 Examples

In this section we give some examples of our general settings, considering first two finite and then two infinite-dimensional cases. Among other aspects, we will see that it is not so rare that some $\varphi_n^{(1)}$ belongs to $\ker(X^\dagger)$, and we will see how this implies that the set of eigenvalues of Θ_2 is just a proper subset of the set of the eigenvalues of Θ_1 , as already mentioned in Section 2.

3.1 A first finite-dimensional example

A first no-go result suggests that, if $\dim(\mathcal{H}) < \infty$, there is no square matrix X such that X is not invertible and still N_2 is strictly positive. This is because, since $\det(X) = 0$, $\det(N_2) = \det(X^\dagger X) = 0$. Then, N_2^{-1} does not exist and the working conditions of Section 2.3 are never satisfied. However, we are left with the possibility of using the approaches discussed in Sections 2.1 or 2.2.

For concreteness, we consider now the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} \\ -\overline{x_{12}} & \overline{x_{11}} \end{pmatrix}$$

With this particularly simple choice, we have $N_1 = N_2 = \tilde{x}\mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix and $\tilde{x} = |x_{11}|^2 + |x_{12}|^2 = \det(X)$. It is clear that, if x_{11} or x_{12} are non zero, N_2^{-1} does exist. Also, $[N_1, \Theta_1] = 0$ for any two-by-two matrix Θ_1 . However, since $\det(X) = \tilde{x} > 0$, it is clear that X^{-1} exists as well. Hence we could adopt the strategies proposed in Sections 2.1 and 2.2. We will not make any explicit choice of Θ_1 here. Rather than this, we observe that $X^{-1} = (\tilde{x})^{-1} X^\dagger$. This implies that $\tilde{\varphi}_n^{(2)} = X^{-1} \varphi_n^{(1)} = (\tilde{x})^{-1} X^\dagger \varphi_n^{(1)} = (\tilde{x})^{-1} \varphi_n^{(2)}$, which is in complete agreement with (2.3).

3.2 A second finite-dimensional example

A possible way out from the previous no-go result consists in considering an operator X between different Hilbert spaces, \mathcal{H}_1 and \mathcal{H}_2 . In this case, if $\dim(\mathcal{H}_1) \neq \dim(\mathcal{H}_2)$, with $\dim(\mathcal{H}_j) < \infty$, $j = 1, 2$, we can still construct examples of the framework discussed in Section 2.3. Let us consider the following non self-adjoint operator Θ_1 , defined on $\mathcal{H}_1 = \mathbb{C}^3$:

$$\Theta_1 = \begin{pmatrix} \frac{1}{6}((5 + \sqrt{3})E_1 - (1 + \sqrt{3})E_2 + 2E_3) & \frac{1}{6}((1 + \sqrt{3})E_1 - (2 + \sqrt{3})E_2 + E_3) & \frac{1}{6}((-7 - 3\sqrt{3})E_1 + (5 + 3\sqrt{3})E_2 + 2E_3) \\ \frac{1}{3}((1 - 2\sqrt{3})E_1 + 2(-1 + \sqrt{3})E_2 + E_3) & \frac{1}{3}(-2E_1 + 4E_2 + E_3) & \frac{1}{3}((1 + 2\sqrt{3})E_1 - 2(1 - \sqrt{3})E_2 + E_3) \\ \frac{1}{6}((-7 + 3\sqrt{3})E_1 + (5 - 3\sqrt{3})E_2 + 2E_3) & \frac{1}{6}((1 - \sqrt{3})E_1 + (-2 + \sqrt{3})E_2 + E_3) & \frac{1}{6}((5 - \sqrt{3})E_1 + (-1 + \sqrt{3})E_2 + 2E_3) \end{pmatrix}$$

with eigenvectors

$$\varphi_1^{(1)} = \begin{pmatrix} -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \varphi_2^{(1)} = \begin{pmatrix} -\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{2}} \\ 2\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \varphi_3^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We have $\Theta_1 \varphi_n^{(1)} = E_n \varphi_n^{(1)}$, $n = 1, 2, 3$. Here we assume that $E_j \in \mathbb{R}$, for all j . We further define

$$X = \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \Rightarrow N_1 = XX^\dagger = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \quad N_2 = X^\dagger X = \frac{3}{2} \mathbb{1}_2,$$

where $\mathbb{1}_2$ is the identity matrix in $\mathcal{H}_2 = \mathbb{C}^2$. Of course N_2^{-1} exists while X^{-1} does not [25]. Moreover, $\Theta_1 N_1 = N_1 \Theta_1$. Hence, we are in the conditions of Section 2.3. Now:

$$\varphi_1^{(2)} = X^\dagger \varphi_1^{(1)} = \begin{pmatrix} \frac{-3+\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2}\sqrt{3(2+\sqrt{3})} \end{pmatrix}, \quad \varphi_2^{(2)} = X^\dagger \varphi_2^{(1)} = \begin{pmatrix} \frac{-6+\sqrt{3}}{2\sqrt{2}} \\ -\frac{3+2\sqrt{3}}{2\sqrt{2}} \end{pmatrix}, \quad \varphi_3^{(2)} = X^\dagger \varphi_3^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows, in particular, that $\varphi_3^{(1)} \in \ker(X^\dagger)$. This is not surprising, since $\dim(\mathcal{H}_2) = 2$, and for this reason we can only have, at most, two linearly independent vectors in \mathcal{H}_2 . We also find, using (2.5),

$$\Theta_2 = \frac{1}{4} \begin{pmatrix} -(1 + \sqrt{3})E_1 + (5 + \sqrt{3})E_2 & (7 - 3\sqrt{3})(E_1 - E_2) \\ -(5 + 3\sqrt{3})(E_1 - E_2) & (5 + \sqrt{3})E_1 - (1 + \sqrt{3})E_2 \end{pmatrix},$$

and we see that E_3 does not appear anymore in Θ_2 . This is in agreement with the fact that $\Theta_2 \varphi_n^{(2)} = E_n \varphi_n^{(2)}$, $n = 1, 2$. Moreover, an explicit computation shows that $X\Theta_2 = \Theta_1 X$ and that $X^\dagger \Theta_1 = \Theta_2 X^\dagger$.

Now the vectors of $\mathcal{F}_\psi^{(1)}$ can be found to be

$$\psi_1^{(1)} = \begin{pmatrix} -\sqrt{2} + \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ \sqrt{2} + \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \psi_2^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3} + \frac{1}{\sqrt{3}}} \end{pmatrix}, \quad \psi_3^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

They are biorthogonal to $\mathcal{F}_\varphi^{(1)}$, $\langle \varphi_n^{(1)}, \psi_m^{(1)} \rangle = \delta_{n,m}$, and, as expected, they are eigenstates of Θ_1^\dagger : $\Theta_1^\dagger \psi_n^{(1)} = E_n \psi_n^{(1)}$, $n = 1, 2, 3$. Now

$$\psi_1^{(2)} = X^\dagger \psi_1^{(1)} = \begin{pmatrix} \sqrt{\frac{3}{2} + \frac{3}{2\sqrt{2}}} \\ \frac{-6+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}, \quad \psi_2^{(2)} = X^\dagger \psi_2^{(1)} = \begin{pmatrix} -\frac{1}{2}\sqrt{3(2+\sqrt{3})} \\ \frac{1}{2}\sqrt{3(2-\sqrt{3})} \end{pmatrix},$$

while we find that

$$\psi_3^{(2)} = X^\dagger \psi_3^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $\psi_3^{(1)} \in \ker(X^\dagger)$, and $\Theta_2^\dagger \psi_n^{(2)} = E_n \psi_n^{(2)}$, $n = 1, 2$. Moreover we get $\langle \varphi_n^{(2)}, \psi_m^{(2)} \rangle = \frac{3}{2} \delta_{n,m}$, $n = 1, 2$. Therefore $\tilde{k}_1 = \tilde{k}_2 = \frac{3}{2}$.

Remark:— It might be interesting to observe that the matrix Θ_2 can be written in terms of pseudo-fermionic operators, see [13]. Two such operators a and b , satisfy $\{a, b\} = \mathbb{1}$, $a^2 = b^2 = 0$, and can be represented for instance as

$$a = \alpha_{12} \begin{pmatrix} \alpha & 1 \\ -\alpha^2 & -\alpha \end{pmatrix}, \quad b = \beta_{12} \begin{pmatrix} \beta & 1 \\ -\beta^2 & -\beta \end{pmatrix},$$

where, calling $\gamma^2 = -\alpha_{12}\beta_{12}$, the parameters must be such that $(\alpha - \beta)^2 \gamma^2 = 1$. In [16] we have seen that the general Hamiltonian $H = \omega ba + \rho \mathbb{1}$ takes the form

$$H = \begin{pmatrix} \omega\gamma\alpha + \rho & \omega\gamma \\ -\omega\gamma\alpha\beta & -\omega\gamma\beta + \rho \end{pmatrix}. \quad (3.1)$$

For any such operator, using some general facts arising from the general pseudo-fermionic structure, we have proposed a simple method, based on ladder operators, to deduce the eigenvalues and eigenvectors of H and of H^\dagger . Intertwining relations can also be deduced, and different scalar products on the Hilbert space \mathbb{C}^2 are shown to play a role, in some cases.

Now, Θ_2 can be written as in (3.1) taking, for instance, $\alpha = -2 - \sqrt{3}$, $\beta = \frac{\sqrt{3}+1}{3\sqrt{3}-7}$, $\rho = E_1$, $\alpha_{12} = \sqrt{\frac{1}{8}(38 - 21\sqrt{3})} = -\beta_{12}$ and $\omega\gamma = \frac{1}{4}(7 - 3\sqrt{3})(E_1 - E_2)$, with γ as above.

Of course, this has useful consequences, as discussed in [13, 16], since the general framework proposed there for the deformed anti commutation relations can now be used for Θ_2 . Viceversa, of course, one could also look now for explicit connections of Θ_2 with some quantum mechanical system, for instance in the context of PT-quantum mechanics.

Finally, since Θ_2 arises out of the 3-by-3 matrix Θ_1 , one can imagine that investigating the inverse map $\Theta_2 \rightarrow \Theta_1$, the pseudo-fermionic framework (which is originally defined in a two-dimensional space) can be somehow extended to a 3-dimensional Hilbert space, similarly to what was done in [17].

3.3 A first example with $\dim(\mathcal{H}) = \infty$

Let $\{\epsilon_n\}$, with $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \dots$, and $\{\theta_n\}$ be two sequences of real numbers, and let us assume that, at least for some n , θ_n is not an integer multiple of π . In this way, $\Im\{e^{i\theta_n}\} \neq 0$. Now, let $\mathcal{F}_e = \{e_n, n \geq 0\}$ be an orthonormal (o.n.) basis for \mathcal{H} , and let us define the following operator:

$$D(\Theta_1) = \left\{ f \in \mathcal{H} : \sum_{n=0}^{\infty} \epsilon_n e^{i\theta_n} \langle e_n, f \rangle e_n \in \mathcal{H} \right\},$$

and

$$\Theta_1 f = \sum_{n=0}^{\infty} \epsilon_n e^{i\theta_n} \langle e_n, f \rangle e_n,$$

for all $f \in D(\Theta_1)$. Of course, $D(\Theta_1)$ is dense since it contains the linear span of the e_n 's, $\mathcal{L}_e = l.s.\{e_n\}$. In particular, $\Theta_1 e_n = \epsilon_n e^{i\theta_n} e_n$, for all n . This shows that $\epsilon_n e^{i\theta_n}$ are the eigenvalues of Θ_1 , not all reals, that $\Theta_1 \neq \Theta_1^\dagger$, and that $\varphi_n^{(1)} = e_n$. This is an explicit example showing that the eigenvectors of a manifestly non self-adjoint operator can still form an o.n. basis. Notice also that we are here considering a sort of *inverse problem*: rather than deducing the eigensystem out of a given operator, we use a given set of numbers and vectors to define an operator which has this particular set of numbers and vectors as eigensystem.

Now, an operator X having the properties required in Section 2.3 can be defined as follows:

$$D(X) = \left\{ f \in \mathcal{H} : \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} \langle e_n, f \rangle e_{n+1} \in \mathcal{H} \right\},$$

which also contains \mathcal{L}_e , and

$$Xf = \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} \langle e_n, f \rangle e_{n+1},$$

for all $f \in D(X)$. The adjoint X^\dagger of X is $X^\dagger g = \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} \langle e_{n+1}, g \rangle e_n$, for all $g \in D(X^\dagger)$, which again contains \mathcal{L}_e . X^\dagger behaves as a lowering operator for \mathcal{F}_e . Indeed we have

$$X^\dagger e_n = \begin{cases} 0, & \text{if } n = 0 \\ \sqrt{\epsilon_n} e_{n-1} & \text{if } n \geq 1. \end{cases}$$

Of course, X is a raising operator: $X e_n = \sqrt{\epsilon_{n+1}} e_{n+1}$. As for N_1 and N_2 , we have $D(N_j) \supseteq \mathcal{L}_e$, $j = 1, 2$, and we find, for all $f \in \mathcal{L}_e$,

$$N_1 f = \sum_{n=0}^{\infty} \epsilon_n \langle e_n, f \rangle e_n, \quad N_2 f = \sum_{n=0}^{\infty} \epsilon_{n+1} \langle e_n, f \rangle e_n.$$

Then, N_2^{-1} does exist, and a direct computation shows that $\Theta_1 N_1 f = N_1 \Theta_1 f$, for all $f \in \mathcal{L}_e$, which is left stable by the action of all the operators introduced so far. Hence, \mathcal{L}_e plays the role of the set \mathcal{D} introduced in Section 2.2, and the assumptions of Section 2.3 are satisfied (in their extended version adapted to infinite dimensional Hilbert spaces).

The set $\mathcal{F}_\psi^{(1)}$ biorthogonal to $\mathcal{F}_\varphi^{(1)} = \mathcal{F}_e$ is, of course, \mathcal{F}_e itself, and equation (2.6) can be explicitly verified. Analogously, all the intertwining equations introduced in Section 2 can be explicitly checked. As for Θ_2 , we find that

$$\Theta_2 f = \sum_{n=1}^{\infty} \epsilon_n e^{i\theta_n} \langle e_{n-1}, f \rangle e_{n-1},$$

for all $f \in D(\Theta_2)$, which is the set of vectors for which the series $\sum_{n=1}^{\infty} \epsilon_n e^{i\theta_n} \langle e_{n-1}, f \rangle e_{n-1}$ converges in \mathcal{H} . Once more, $\mathcal{L}_e \subseteq D(\Theta_2)$. Now, since

$$\varphi_n^{(2)} = X^\dagger \varphi_n^{(1)} = X^\dagger e_n = \begin{cases} 0, & \text{if } n = 0 \\ \sqrt{\epsilon_n} e_{n-1} & \text{if } n \geq 1, \end{cases}$$

we can check that $\Theta_2 \varphi_n^{(2)} = \epsilon_n \varphi_n^{(2)}$, for all $n \geq 0$. Moreover, since $\psi_n^{(1)} = \varphi_n^{(1)} = e_n$, and since $\psi_n^{(2)} = X^\dagger \psi_n^{(1)}$, it follows that $\psi_n^{(2)} = \varphi_n^{(2)}$. We find that $\Theta_2^\dagger \psi_n^{(2)} = \overline{\epsilon_n} \psi_n^{(2)}$, as well as $N_j \varphi_n^{(j)} = \epsilon_n \varphi_n^{(j)}$ and $N_j \psi_n^{(j)} = \epsilon_n \psi_n^{(j)}$, $j = 1, 2$. Finally, while $\langle \varphi_n^{(1)}, \psi_m^{(1)} \rangle = \delta_{n,m}$, we have $\langle \varphi_n^{(2)}, \psi_m^{(2)} \rangle = \epsilon_n \delta_{n,m}$. Hence the general structure of Section 2.3 is fully recovered. It might be interesting to notice that the orthogonality of the sets $\mathcal{F}_\varphi^{(j)}$ and $\mathcal{F}_\psi^{(j)}$, $j = 1, 2$, was expected because their vectors are eigenstates of N_1 and N_2 with eigenvalues which are not degenerate.

3.4 A second example with $\dim(\mathcal{H}) = \infty$

This example adapts what was originally discussed in [15] to the present situation. Let Θ_1 be the following (infinite) matrix, acting on the Hilbert space $\mathcal{H} = l^2(\mathbb{N})$:

$$\Theta_1 = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 & . & . \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & \alpha_2 & \beta_2 & 0 & 0 & . & . \\ 0 & 0 & \beta_2 & \alpha_2 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \alpha_3 & \beta_3 & . & . \\ 0 & 0 & 0 & 0 & \beta_3 & \alpha_3 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

where α_j and β_j are, in general, complex numbers. Its eigenvectors are

$$\varphi_1^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \end{pmatrix}, \varphi_2^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \end{pmatrix}, \varphi_3^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ . \\ . \end{pmatrix}, \varphi_4^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ . \\ . \end{pmatrix},$$

and so on. The corresponding eigenvalues are $\epsilon_1 = \alpha_1 - \beta_1$, $\epsilon_2 = \alpha_1 + \beta_1$, $\epsilon_3 = \alpha_2 - \beta_2$, $\epsilon_4 = \alpha_2 + \beta_2$, \dots : $\Theta_1 \varphi_n^{(1)} = \epsilon_n \varphi_n^{(1)}$. It is clear that, fixing properly α_j and β_j , all the eigenvalues turn out to be different. Also, if the imaginary parts of some α_j or β_j is non zero, then not all the eigenvalues of Θ_1 are real. It is also important to notice that Θ_1 is densely defined, since it is well defined on the linear span of the $\varphi_n^{(1)}$'s, which form an o.n. basis for \mathcal{H} .

Remarks:– (1) If each β_j is purely imaginary and each α_j is real, then it is clear that $\epsilon_{2n-1} = \overline{\epsilon_{2n}}$, for all $n \geq 1$. Then we can think of Θ_1 as a non self-adjoint Hamiltonian in a broken phases, in which the eigenvalues (which are real in the unbroken region) become conjugate in pairs. With this in mind, this example could be relevant in the context of PT quantum mechanics.

(2) In view of the Remark in Section 3.2 it is interesting to notice that we can introduce a pseudo-fermionic structure even here, defining a pair of operators a and b satisfying $\{a, b\} = \mathbb{1}$

and $a^2 = b^2 = 0$, for each submatrix

$$[\Theta_1]_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j & \alpha_j \end{pmatrix},$$

which can be rewritten as in (3.1). For this, it is sufficient to take $\omega = 2\beta_j$, $\alpha = -\beta = 1$, $\rho = \alpha_j - \beta_j$ and $\alpha_{12} = -\beta_{12} = \frac{1}{2}$.

Let now introduce a bounded operator X , see [15]:

$$\mathcal{H} \ni f \rightarrow Xf = \{(Xf)_j, j \in \mathbb{N}\} = \{\langle \eta_j, f \rangle, j \in \mathbb{N}\},$$

where $\mathcal{F}_\eta = \{\eta_j\}$ is a tight frame of \mathcal{H} defined as follows: $\eta_{2n-1} = \eta_{2n} = \frac{1}{\sqrt{2}}e_n$, $n \geq 1$. Here e_n is the n -th vector of the o.n. canonical basis of $l^2(\mathbb{N})$, the one with all zero entries except the n -th one, which is equal to one. Then we have $N_2 = X^\dagger X = \mathbb{1}$, which is of course bounded and invertible, while

$$N_1 = XX^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 1 & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 1 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 1 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & 1 & 1 & . & . \\ 0 & 0 & 0 & 0 & 1 & 1 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix} = \frac{1}{2}(\mathbb{1} + P),$$

Here $\mathbb{1}$ is the identity operator on \mathcal{H} , and P is a permutation operator acting as follows:

$$Pc = P \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ . \\ . \end{pmatrix} = \begin{pmatrix} c_2 \\ c_1 \\ c_4 \\ c_3 \\ c_6 \\ c_5 \\ . \\ . \end{pmatrix},$$

for all $c \in \mathcal{H}$. N_1 is also bounded. We see that $[N_1, \Theta_1] = 0$, so that we are in the conditions of Section 2.3. A simple computation shows that $\Theta_2 = N_2^{-1}(X^\dagger \Theta_1 X) = X^\dagger \Theta_1 X$ is a

diagonal matrix, with elements $\alpha_k + \beta_k$. In bra-ket notation: $\Theta_2 = \sum_{k=1}^{\infty} (\alpha_k + \beta_k) |e_k\rangle\langle e_k| = \sum_{k=1}^{\infty} \epsilon_k |e_k\rangle\langle e_k|$. It is evident that all the *original odd eigenvalues* ϵ_{2n-1} disappear from the game! The reason is simple: each $\varphi_{2n-1}^{(1)}$ belongs to the kernel of X^\dagger . Hence $\varphi_{2n-1}^{(2)} = 0$. On the other hand, $\varphi_{2n}^{(2)} = X^\dagger \varphi_{2n}^{(1)} = e_n$, for all n . Then the set of the eigenvalues of Θ_2 is properly contained in the set of the eigenvalues of Θ_1 , but the set of eigenvectors is still a basis for \mathcal{H} .

Remark:— If β_j and α_j are real, and if $\beta_j > \alpha_j > 0$, for all j , then it is clear that each $\epsilon_{2n} > 0$, while each $\epsilon_{2n-1} < 0$, for all $n \geq 1$. Then, going from Θ_1 to Θ_2 can be seen as a sort of *filter*, which removes all the negative eigenvalues present in the original operator Θ_1 .

It is a simple exercise to check that the intertwining relations $X\Theta_2 = \Theta_1 X$ and $X^\dagger\Theta_1 = \Theta_2 X^\dagger$ hold true.

Finding now the set $\mathcal{F}_\psi^{(1)}$ is quite easy, due to the fact that $\mathcal{F}_\varphi^{(1)}$ is already an o.n. set. Hence $\psi_n^{(1)} = \varphi_n^{(1)}$, for all n . It is also easy to check that $\Theta_1^\dagger \psi_n^{(1)} = \overline{\epsilon_n} \psi_n^{(1)}$. As for Θ_2^\dagger we get $\Theta_2^\dagger = \sum_{k=1}^{\infty} (\overline{\alpha_k} + \overline{\beta_k}) |e_k\rangle\langle e_k|$, and $\psi_n^{(2)} = \varphi_n^{(2)}$. We will briefly return to this example in Section 4, in connection with bicoherent states.

3.4.1 A remark, more than an example: nonlinear \mathcal{D} -pseudo bosons

In some recent paper the notion of nonlinear \mathcal{D} -pseudo bosons has been introduced and analyzed in some details, [18, 19, 20]. An output of this notion is that the eigenvalues and the eigenvectors of the factorized operators $M = ba$ and $M^\dagger = a^\dagger b^\dagger$ can be deduced using some minimal and natural assumptions on a and b . Here a and b , and their adjoints, are suitable raising and lowering operators on different sets of vectors. More in details, let us consider a strictly increasing sequence $\{\epsilon_n\}$: $0 = \epsilon_0 < \epsilon_1 < \dots < \epsilon_n < \dots$. Then, given two operators a and b on \mathcal{H} , and a set $\mathcal{D} \subset \mathcal{H}$ which is dense in \mathcal{H} , and which is stable under the action of a, b, a^\dagger and b^\dagger ,

Definition 3.1 *We will say that the triple $(a, b, \{\epsilon_n\})$ is a family of \mathcal{D} -non linear pseudo-bosons (\mathcal{D} -NLPBs) if the following properties hold:*

- **p1.** *a non zero vector Φ_0 exists in \mathcal{D} such that $a\Phi_0 = 0$;*
- **p2.** *a non zero vector η_0 exists in \mathcal{D} such that $b^\dagger\eta_0 = 0$;*
- **p3.** *Calling*

$$\Phi_n := \frac{1}{\sqrt{\epsilon_n!}} b^n \Phi_0, \quad \eta_n := \frac{1}{\sqrt{\epsilon_n!}} a^{\dagger n} \eta_0, \quad (3.2)$$

we have, for all $n \geq 0$,

$$a\Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \quad b^\dagger\eta_n = \sqrt{\epsilon_n} \eta_{n-1}. \quad (3.3)$$

- **p4.** The set $\mathcal{F}_\Phi = \{\Phi_n, n \geq 0\}$ is a basis for \mathcal{H} .

Of course, since \mathcal{D} is stable under the action of b and a^\dagger , it follows that $\Phi_n, \eta_n \in \mathcal{D}$, for all $n \geq 0$. The set $\mathcal{F}_\eta = \{\eta_n, n \geq 0\}$ is a basis for \mathcal{H} as well. This follows from the fact that $M\Phi_n = \epsilon_n \Phi_n$ and $M^\dagger \eta_n = \epsilon_n \eta_n$. Therefore, choosing the normalization of η_0 and Φ_0 in such a way $\langle \eta_0, \Phi_0 \rangle = 1$, \mathcal{F}_η is biorthogonal to the basis \mathcal{F}_Φ . Then, it is possible to check that \mathcal{F}_η is the unique basis which is biorthogonal to \mathcal{F}_Φ .

We are apparently in the situation considered in Section 2.3, with $\Theta_1 = M = ba$, $\varphi_n^{(1)} = \Phi_n$, $X = b$ and $\Theta_2 = ab$. In fact, with these choices we have $\Theta_1 X = X \Theta_2$, which is what is required first in Proposition 2.2. Moreover, the eigenvectors $\psi_n^{(1)}$ must be identified with the η_n 's [28]. However, Proposition 2.2 cannot be applied here. In fact: first of all b does not admit inverse, so that point (i) of the cited Proposition does not apply. Moreover, there is no reason for bb^\dagger to commute with Θ_1 , and for this reason also point (ii) cannot be used in its most relevant part (the construction of the eigenvectors of Θ_2).

However, for completeness, we will now briefly see that we can still use a different approach, deducing in this way the eigenvectors of many relevant operators involved in the game, but unfortunately nothing particularly interesting arise. In fact,

$$\Theta_2 (a\varphi_n^{(1)}) = a \Theta_1 \varphi_n^{(1)} = \epsilon_n (a\varphi_n^{(1)}) .$$

Then $\varphi_n^{(2)} = a\varphi_n^{(1)}$ which is, because of **p3.** above, equal to $\sqrt{\epsilon_n} \varphi_{n-1}^{(1)}$, for all $n \geq 0$. Here we are simply putting $\varphi_{-1}^{(1)} = 0$. Similarly we see that $b^\dagger \psi_n^{(1)}$ are the eigenstates of Θ_2^\dagger . But $b^\dagger \psi_n^{(1)}$ coincides with $\sqrt{\epsilon_n} \psi_{n-1}^{(1)}$, again with the agreement that $\psi_{-1}^{(1)} = 0$. In conclusion, in this case, we can still easily find the eigenstates and the eigenvalues of Θ_j and Θ_j^\dagger , but nothing particularly new appears going from $(\Theta_1, \Theta_1^\dagger)$ to $(\Theta_2, \Theta_2^\dagger)$: the eigenvectors are the same, but they are just *shifted* in their quantum number. This is not really particularly unexpected, since it is a frequent feature in supersymmetric quantum mechanics, where factorizable Hamiltonians play a relevant role, [21].

4 Bicoherent states

We will now discuss how coherent states can be defined within the general framework considered in Section 2, and which kind of properties do they have. The working assumptions in this section are the following: (1) $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\psi^{(1)}$ are biorthogonal bases for \mathcal{H} , which will be considered in this section to be infinite-dimensional; (2) the eigenvalues ϵ_n of Θ_1 are positive. In particular,

without loss of generality, we can always assume that $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \dots$, as it is usually considered in the literature on (generalized) coherent states, see [22] and references therein. Notice that this condition was already assumed in this paper, in Sections 3.3 and 3.4.1.

Under our assumptions the sets $\mathcal{L}_\varphi^{(1)} = l.s.\{\varphi_n^{(1)}\}$ and $\mathcal{L}_\psi^{(1)} = l.s.\{\psi_n^{(1)}\}$ are dense in \mathcal{H} . We define two operators A_1 and B_1^\dagger as follows:

$$\mathcal{L}_\varphi^{(1)} \ni f = \sum_{k=0}^N c_k \varphi_k^{(1)}, \quad \Rightarrow \quad A_1 f = \sum_{k=1}^N c_k \sqrt{\epsilon_k} \varphi_{k-1}^{(1)},$$

and

$$\mathcal{L}_\psi^{(1)} \ni g = \sum_{k=0}^M d_k \psi_k^{(1)}, \quad \Rightarrow \quad B_1^\dagger g = \sum_{k=1}^M d_k \sqrt{\epsilon_k} \psi_{k-1}^{(1)},$$

for $M, N < \infty$. Then, in particular we have

$$A_1 \varphi_k^{(1)} = \begin{cases} 0, & \text{if } k = 0 \\ \sqrt{\epsilon_k} \varphi_{k-1}^{(1)} & \text{if } k \geq 1, \end{cases} \quad \text{and} \quad B_1^\dagger \psi_k^{(1)} = \begin{cases} 0, & \text{if } k = 0 \\ \sqrt{\epsilon_k} \psi_{k-1}^{(1)} & \text{if } k \geq 1. \end{cases} \quad (4.1)$$

From these formulas, using the biorthogonality and the completeness of $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\psi^{(1)}$, we also deduce that

$$A_1^\dagger \psi_k^{(1)} = \sqrt{\epsilon_{k+1}} \psi_{k+1}^{(1)}, \quad B_1 \varphi_k^{(1)} = \sqrt{\epsilon_{k+1}} \varphi_{k+1}^{(1)}, \quad (4.2)$$

for all $k \geq 0$. Hence A_1 and B_1^\dagger act as lowering operators, while A_1^\dagger and B_1 behave as raising operators, on different sets[?]. A consequence of these equations is that both Θ_1 and Θ_1^\dagger can be factorized in the following way:

$$\Theta_1 \varphi_n^{(1)} = B_1 A_1 \varphi_n^{(1)} = \epsilon_n \varphi_n^{(1)}, \quad \Theta_1^\dagger \psi_n^{(1)} = A_1^\dagger B_1^\dagger \psi_n^{(1)} = \epsilon_n \psi_n^{(1)}, \quad (4.3)$$

for all $n \geq 0$.

The following Proposition shows how bicoherent states can be introduced, and which are their properties.

Proposition 4.1 *Let us assume that there exist four constants $r_\varphi, r_\psi > 0$, and $0 \leq \alpha_\varphi, \alpha_\psi \leq \frac{1}{2}$, such that $\|\varphi_n^{(1)}\| \leq r_\varphi^n (\epsilon_n!)^{\alpha_\varphi}$ and $\|\psi_n^{(1)}\| \leq r_\psi^n (\epsilon_n!)^{\alpha_\psi}$, for all $n \geq 0$. Let us define*

$$\rho_\varphi = \frac{1}{r_\varphi} \lim_k (\epsilon_{k+1})^{1/2-\alpha_\varphi}, \quad \rho_\psi = \frac{1}{r_\psi} \lim_k (\epsilon_{k+1})^{1/2-\alpha_\psi}, \quad \hat{\rho} = \lim_k \epsilon_{k+1},$$

and $\rho := \min(\rho_\varphi, \rho_\psi, \sqrt{\hat{\rho}})$. Let $C_\rho(0)$ be the circle in the complex plane centered in the origin and with radius ρ . Then, defining

$$N(|z|) = \left(\sum_{k=0}^{\infty} \frac{|z|^{2k}}{\epsilon_k!} \right)^{-1/2}, \quad (4.4)$$

and

$$\varphi_1(z) = N(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\epsilon_k!}} \varphi_k^{(1)}, \quad \psi_1(z) = N(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\epsilon_k!}} \psi_k^{(1)}, \quad (4.5)$$

these are all well defined for $z \in C_\rho(0)$. Moreover, for all such z , $\langle \varphi_1(z), \psi_1(z) \rangle = 1$, $A_1 \varphi_1(z) = z \varphi_1(z)$ and $B_1^\dagger \psi_1(z) = z \psi_1(z)$. Also, if a measure $d\lambda(r)$ exists such that $\int_0^\rho d\lambda(r) r^{2k} = \frac{\epsilon_k!}{2\pi}$, for all $k \geq 0$, then, calling $d\nu(z, \bar{z}) = d\lambda(r) d\theta$, we have

$$\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} \langle f, \varphi_1(z) \rangle \langle \psi_1(z), g \rangle = \langle f, g \rangle, \quad (4.6)$$

for all $f, g \in \mathcal{H}$.

Proof – First of all it is clear that the series in (4.4) converges in $C_\rho(0)$, since it converges if $|z|^2 < \hat{\rho}$. Now, let us compute $\|\varphi_1(z)\|^2$. Using our assumption on $\|\varphi_n^{(1)}\|$ we get

$$\|\varphi_1(z)\| \leq |N(|z|)| \sum_{k=0}^{\infty} \frac{|z|^k}{\sqrt{\epsilon_k!}} \|\varphi_k^{(1)}\| \leq |N(|z|)| \sum_{k=0}^{\infty} \frac{(|z|r_\varphi)^k}{(\epsilon_k!)^{1/2-\alpha_\varphi}}.$$

The power series in the right-hand side converges for all z with $|z| < \rho_\varphi$. Analogously, we can prove that $\|\psi_1(z)\|$ is bounded from above by a power series (times $|N(|z|)|$), which converges for all z with $|z| < \rho_\psi$. Concluding, if $|z| < \rho$, all the relevant power series do converge: hence both $\varphi_1(z)$ and $\psi_1(z)$ are well defined. Moreover, it is easy to check that $\langle \varphi_1(z), \psi_1(z) \rangle = 1$ for all such z 's.

To show that $\varphi_1(z)$ is an eigenstate of A_1 we use (4.1):

$$\begin{aligned} A_1 \varphi_1(z) &= N(|z|) A_1 \left(\varphi_0^{(1)} + \frac{z}{\epsilon_1!} \varphi_1^{(1)} + \frac{z^2}{\epsilon_2!} \varphi_2^{(1)} + \frac{z^3}{\epsilon_3!} \varphi_3^{(1)} + \dots \right) = \\ &= z N(|z|) \left(\varphi_0^{(1)} + \frac{z}{\epsilon_1!} \varphi_1^{(1)} + \frac{z^2}{\epsilon_2!} \varphi_2^{(1)} + \frac{z^3}{\epsilon_3!} \varphi_3^{(1)} + \dots \right) = z \varphi_1(z). \end{aligned}$$

In the same way, using the lowering properties of B_1^\dagger on the vectors $\psi_n^{(1)}$, we deduce that $B_1^\dagger \psi_1(z) = z \psi_1(z)$.

Finally, to check the resolution of the identity in (4.6), we observe that, taken $f, g \in \mathcal{H}$,

$$\begin{aligned} &\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} \langle f, \varphi_1(z) \rangle \langle \psi_1(z), g \rangle = \\ &= \sum_{k,l=0}^{\infty} \frac{\langle f, \varphi_k^{(1)} \rangle \langle \psi_l^{(1)}, g \rangle}{\sqrt{\epsilon_k! \epsilon_l!}} \int_0^\rho d\lambda(r) r^k r^l \int_0^{2\pi} d\theta e^{ik\theta} e^{-il\theta} = \end{aligned}$$

$$\begin{aligned}
&= 2\pi \sum_{k=0}^{\infty} \frac{\langle f, \varphi_k^{(1)} \rangle \langle \psi_k^{(1)}, g \rangle}{\epsilon_k!} \int_0^\rho d\lambda(r) r^{2k} = \\
&= \sum_{k=0}^{\infty} \langle f, \varphi_k^{(1)} \rangle \langle \psi_k^{(1)}, g \rangle = \langle f, g \rangle.
\end{aligned}$$

We have used here the property of the measure $d\lambda(r)$, and the fact that $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\psi^{(1)}$ are biorthogonal bases. □

Remarks:– (1) This Proposition extends significantly a similar result originally given in [24], where only the existence of states similar to our $\varphi_1(z)$ and $\psi_1(z)$ was discussed.

(2) It is clear that we also have

$$\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} |\psi_1(z)\rangle \langle \varphi_1(z)| = \mathbb{1},$$

which is analogous to (4.6) but expressed in terms of Dirac bras and kets.

(3) The existence of ρ_φ , ρ_ψ and $\hat{\rho}$ is not guaranteed a priori. But it is clear that do exist in some particular cases. For instance, when our framework collapses to the one of *standard* coherent states, [22], i.e. when $B_1 = A_1^\dagger$, $\epsilon_k = k$ and when $\mathcal{F}_\varphi^{(1)}$ coincides with $\mathcal{F}_\psi^{(1)}$ and is an orthonormal basis. Then we can take $\alpha_\varphi = \alpha_\psi = 0$ and $r_\varphi = r_\psi = 1$, so that $\rho_\varphi = \rho_\psi = \hat{\rho} = \infty$. Hence we have convergence in all the complex plane. Another example is discussed in Section 4.2. Other examples can be found in [24].

So far we have considered bicoherent states constructed using the eigenstates of Θ_1 and of Θ_1^\dagger . In Section 2 we have shown how, to these operators and these eigenvectors, we can associate new operators (Θ_2 and Θ_2^\dagger) and new eigenvectors, $\varphi_k^{(2)}$ and $\psi_k^{(2)}$. It is natural, therefore, to check if also these vectors can be used to define a new pair of bicoherent states, and how.

Before starting, we remind that while $\langle \varphi_n^{(1)}, \psi_k^{(1)} \rangle = \delta_{n,k}$, $\langle \varphi_k^{(2)}, \psi_n^{(2)} \rangle = \tilde{k}_n \delta_{k,n}$, see (2.10). This fact has consequences in the definition of $\varphi_2(z)$ and $\psi_2(z)$, which we now introduce as follows

$$\varphi_2(z) = N(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\epsilon_k! \tilde{k}_k}} \varphi_k^{(2)}, \quad \psi_2(z) = N(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\epsilon_k! \tilde{k}_k}} \psi_k^{(2)}. \quad (4.7)$$

We are assuming, for the moment, that all the \tilde{k}_n 's are different from zero. Let us now observe that $\|\varphi_n^{(2)}\|^2 = \|X^\dagger \varphi_n^{(1)}\|^2 = \tilde{k}_n \|\varphi_n^{(1)}\|^2$ and $\|\psi_n^{(2)}\|^2 = \|X^\dagger \psi_n^{(1)}\|^2 = \tilde{k}_n \|\psi_n^{(1)}\|^2$. Then, under the assumptions of Proposition 4.1, with these definitions, both $\varphi_2(z)$ and $\psi_2(z)$ are well defined for

all $z \in C_\rho(0)$. Moreover, once again, $\langle \varphi_2(z), \psi_2(z) \rangle = 1$. If the measure $d\lambda(r)$ is now replaced by a new measure, $d\tilde{\lambda}(r)$, satisfying the new moment problem $\int_0^\rho d\tilde{\lambda}(r) r^{2k} = \frac{\epsilon_k \tilde{k}_k}{2\pi}$, for all $k \geq 0$, then

$$\int_{C_\rho(0)} d\tilde{\nu}(z, \bar{z}) N(|z|)^{-2} |\varphi_2(z)\rangle \langle \psi_2(z)| = \sum_{k=0}^{\infty} |\varphi_k^{(2)}\rangle \langle \psi_k^{(2)}|, \quad (4.8)$$

where $d\tilde{\nu}(z, \bar{z}) = d\tilde{\lambda}(r) d\theta$. Notice that, in general, this is not necessarily equal to the identity operator, except if $\mathcal{F}_\varphi^{(2)}$ and $\mathcal{F}_\psi^{(2)}$ are also bases, which is not granted a priori. It might happen, for instance, that $\mathcal{F}_\varphi^{(2)}$ and $\mathcal{F}_\psi^{(2)}$ are not bases, but they are complete in \mathcal{H} . This is, for instance, what is observed in several applications involving the so-called \mathcal{D} -pseudo bosons, [13], where several sets of eigenvectors of non self-adjoint operators turn out to be complete in \mathcal{H} , but not bases[?] for \mathcal{H} . In both cases, the linear span of the $\varphi_n^{(2)}$'s, $\mathcal{L}_\varphi^{(2)}$, and of the $\psi_n^{(2)}$'s, $\mathcal{L}_\psi^{(2)}$, are again dense in \mathcal{H} . For this reason we can still introduce two (in general, unbounded) operators on these (dense) sets. Again, the difference of mutual normalization between $(\mathcal{F}_\varphi^{(1)}, \mathcal{F}_\psi^{(1)})$ and $(\mathcal{F}_\varphi^{(2)}, \mathcal{F}_\psi^{(2)})$, suggests to change a little bit the definition originally given for A_1 and B_1^\dagger . In fact, it is now convenient to define

$$\mathcal{L}_\varphi^{(2)} \ni f = \sum_{k=0}^N c_k \varphi_k^{(2)}, \quad \Rightarrow \quad A_2 f = \sum_{k=1}^N c_k \sqrt{\frac{\epsilon_k \tilde{k}_k}{\tilde{k}_{k-1}}} \varphi_{k-1}^{(2)},$$

and

$$\mathcal{L}_\psi^{(2)} \ni g = \sum_{k=0}^M d_k \psi_k^{(2)}, \quad \Rightarrow \quad B_2^\dagger g = \sum_{k=1}^M d_k \sqrt{\frac{\epsilon_k \tilde{k}_k}{\tilde{k}_{k-1}}} \psi_{k-1}^{(2)},$$

so that, in particular,

$$A_2 \varphi_k^{(2)} = \begin{cases} 0, & \text{if } k = 0 \\ \sqrt{\frac{\epsilon_k \tilde{k}_k}{\tilde{k}_{k-1}}} \varphi_{k-1}^{(2)} & \text{if } k \geq 1, \end{cases} \quad \text{and} \quad B_2^\dagger \psi_k^{(2)} = \begin{cases} 0, & \text{if } k = 0 \\ \sqrt{\frac{\epsilon_k \tilde{k}_k}{\tilde{k}_{k-1}}} \psi_{k-1}^{(2)} & \text{if } k \geq 1. \end{cases} \quad (4.9)$$

Now, it is obvious that $A_2 \varphi_2(z) = z \varphi_2(z)$ and $B_2^\dagger \psi_2(z) = z \psi_2(z)$, which means that these states are eigenstates of some suitable lowering operators. Summarizing, for $\varphi_2(z)$ and $\psi_2(z)$ we can deduce the same conclusions as in Proposition 4.1, except that (4.6) must be replaced by equation (4.8).

Using A_2 , B_2^\dagger and their adjoints it is also possible to see that Θ_2 and Θ_2^\dagger admit a sort of factorization. In fact, we can see that

$$A_2^\dagger \psi_k^{(2)} = \sqrt{\frac{\epsilon_{k+1} \tilde{k}_k}{\tilde{k}_{k+1}}} \psi_{k+1}^{(2)}, \quad B_2 \varphi_k^{(2)} = \sqrt{\frac{\epsilon_{k+1} \tilde{k}_k}{\tilde{k}_{k+1}}} \varphi_{k+1}^{(2)}, \quad (4.10)$$

for all $k \geq 0$. Therefore, in analogy with equation (4.3), we find that

$$\Theta_2 \varphi_n^{(2)} = B_2 A_2 \varphi_n^{(2)} = \epsilon_n \varphi_n^{(2)}, \quad \Theta_2^\dagger \psi_n^{(2)} = A_2^\dagger B_2^\dagger \psi_n^{(2)} = \epsilon_n \psi_n^{(2)}, \quad (4.11)$$

for all $n \geq 0$.

Remark:— As already mentioned, in what we have done so far, it is important the fact that each $\varphi_n^{(1)} \notin \ker(X^\dagger)$, so that $\varphi_n^{(2)} \neq 0$ and $\tilde{k}_n \neq 0$. In fact, if this is not so and if for some index \hat{n} we have $\tilde{k}_{\hat{n}} = 0$, then many of the previous formulas have problems, and $\varphi_2(z)$ and $\psi_2(z)$ cannot be defined, apparently. This is not really so: let us call $\mathfrak{K} = \{n \in \mathbb{N}_0 : \varphi_n^{(1)} \in \ker(X^\dagger)\}$, and $\mathfrak{K}^c = \mathbb{N}_0 \setminus \mathfrak{K}$. It is useful (but not mandatory) to assume that $0 \in \mathfrak{K}^c$. Of course, \mathfrak{K}^c is a fully ordered set of natural numbers (with zero), so we can relabel the vectors $\varphi_n^{(2)}$'s and the ϵ_n 's in an unique way: $\tilde{\varphi}_l^{(2)} = \varphi_{n_l}^{(2)}$, $\tilde{\epsilon}_l = \epsilon_{n_l}$, for all $l \in \mathbb{N}_0$. Here $0 = n_0 < n_1 < n_2 < n_3 < \dots$. Then we introduce the set $\mathcal{F}_{\tilde{\varphi}}^{(2)} = \{\tilde{\varphi}_l^{(2)}\}$ and, in the same way, a second set $\mathcal{F}_{\tilde{\psi}}^{(2)} = \{\tilde{\psi}_l^{(2)}\}$, extracted out of $\mathcal{F}_{\psi}^{(2)}$. Since $n_k = n_l$ if and only if $k = l$, we deduce that $\mathcal{F}_{\tilde{\varphi}}^{(2)}$ and $\mathcal{F}_{\tilde{\psi}}^{(2)}$ are biorthogonal, $\langle \tilde{\varphi}_l^{(2)}, \tilde{\psi}_n^{(2)} \rangle = \tilde{k}_l \delta_{l,n}$, with all the \tilde{k}_l which are now, by construction, different from zero. Then our new bicoherent states, which replace those in (4.7), are the following:

$$\tilde{\varphi}_2(z) = \tilde{N}(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\tilde{\epsilon}_k! \tilde{k}_k}} \tilde{\varphi}_k^{(2)}, \quad \tilde{\psi}_2(z) = \tilde{N}(|z|) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\tilde{\epsilon}_k! \tilde{k}_k}} \tilde{\psi}_k^{(2)}. \quad (4.12)$$

Nothing really changes with respect with what we have done before. The only difference has to do with the nature of the sets $\mathcal{F}_{\tilde{\varphi}}^{(2)}$ and $\mathcal{F}_{\tilde{\psi}}^{(2)}$, which might no longer be complete (or basis), even when $\mathcal{F}_{\varphi}^{(2)}$ and $\mathcal{F}_{\psi}^{(2)}$ are complete (or basis). The existence of these vectors, and their other properties, can now still be deduced without major differences with respect to what we have shown before.

4.1 Quantization via bicoherent states

The problem of quantizing a classical system has a very long story, and can be approached in several ways. We refer to [22, 23] for many details on this topic. What we want to do here is to show that bicoherent states can also be used for this purpose. In particular we will show that A_1 and B_1 in (4.1) and (4.2) can be seen as the result of a *suitable quantization* of z and \bar{z} . More explicitly, we will now prove that

$$\langle f, A_1 g \rangle = \left\langle f, \left(\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} z |\varphi_1(z) \rangle \langle \psi_1(z)| \right) g \right\rangle, \quad (4.13)$$

and

$$\langle f, B_1 g \rangle = \left\langle f, \left(\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} \bar{z} |\varphi_1(z) \rangle \langle \psi_1(z)| \right) g \right\rangle, \quad (4.14)$$

for all $f \in \mathcal{H}$ and $g \in D(A_1) \cap D(B_1)$, which is dense in \mathcal{H} since it contains $\mathcal{L}_\varphi^{(1)}$.

To prove (4.13), we observe first that, because of the property of $d\lambda(r)$ in $d\nu(z, \bar{z}) = d\lambda(r) d\theta$,

$$\int_{C_\rho(0)} d\nu(z, \bar{z}) z^{k+1} \bar{z}^l = \epsilon_l! \delta_{k+1, l}.$$

Therefore, after some computations very similar to those needed to prove the resolution of the identity,

$$\left\langle f, \left(\int_{C_\rho(0)} d\nu(z, \bar{z}) N(|z|)^{-2} z |\varphi_1(z) \rangle \langle \psi_1(z)| \right) g \right\rangle = \sum_{k=0}^{\infty} \sqrt{\epsilon_{k+1}} \langle f, \varphi_k^{(1)} \rangle \langle \psi_{k+1}^{(1)}, g \rangle, \quad (4.15)$$

which is equal to $\langle f, A_1 g \rangle$. In fact, since $\mathcal{F}_\varphi^{(1)}$ and $\mathcal{F}_\psi^{(1)}$ are biorthogonal bases for \mathcal{H} , we can write $g = \sum_{k=0}^{\infty} \langle \psi_k^{(1)}, g \rangle \varphi_k^{(1)}$. Then, recalling that $g \in D(A_1)$,

$$A_1 g = \sum_{k=0}^{\infty} \langle \psi_k^{(1)}, g \rangle A_1 \varphi_k^{(1)} = \sum_{k=1}^{\infty} \langle \psi_k^{(1)}, g \rangle \sqrt{\epsilon_k} \varphi_{k-1}^{(1)} = \sum_{k=0}^{\infty} \langle \psi_{k+1}^{(1)}, g \rangle \sqrt{\epsilon_{k+1}} \varphi_k^{(1)},$$

which returns the RHS of (4.15) after taking the scalar product with f . Formula (4.14) can be proved in a similar way.

To get the operators A_1^\dagger and B_1^\dagger we can repeat the same steps, but with the role of the φ 's and ψ 's exchanged. Of course, more operators could be written in terms of z , \bar{z} , and functions of these.

4.2 An example

What we are going to discuss here is based on the example discussed in Section 3.4. In particular, we will adopt the following choice: $\beta_k = \alpha_1$ and $\alpha_k = (4k - 3)\alpha_1$, for all $k = 1, 2, 3, \dots$. We also fix $\alpha_1 \in \mathbb{R}$. Hence the eigenvalues of Θ_1 become $\hat{\epsilon}_n = \epsilon_{n+1} = 2n\alpha_1$, $n \geq 0$. The reason for introducing these $\hat{\epsilon}_n$, with the property that $\hat{\epsilon}_0 = 0$, and the related $\hat{\varphi}_n^{(1)} = \varphi_{n+1}^{(1)}$, $n = 0, 1, 2, 3, \dots$, is that this was required at the beginning of this section, and was used in the derivation of some formulas. Of course, since $\hat{\psi}_n^{(1)} = \hat{\varphi}_n^{(1)}$, the bicoherent states $\varphi_1(z)$ and $\psi_1(z)$ coincide:

$$\varphi_1(z) = \psi_1(z) = N_1(|z|) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\hat{\epsilon}_n!}} \hat{\varphi}_n^{(1)}. \quad (4.16)$$

Now, since $\hat{\epsilon}_n! = (2\alpha_1)^n n!$, we deduce that $N_1(|z|) = e^{-\frac{|z|^2}{4\alpha_1}}$, for all $z \in \mathbb{C}$. Formula (4.16) produces now

$$\varphi_1(z) = \psi_1(z) = \frac{1}{\sqrt{2}} e^{-\frac{|z|^2}{4\alpha_1}} \begin{pmatrix} 1 + \frac{z}{\sqrt{2\alpha_1}} \\ -1 + \frac{z}{\sqrt{2\alpha_1}} \\ \frac{z^2}{2\alpha_1\sqrt{2!}} \left(1 + \frac{z}{\sqrt{2\alpha_1}3!}\right) \\ \frac{z^2}{2\alpha_1\sqrt{2!}} \left(-1 + \frac{z}{\sqrt{2\alpha_1}3!}\right) \\ \dots \\ \dots \\ \dots \\ \dots \end{pmatrix}.$$

Since $\|\varphi_k^{(1)}\| = \|\psi_k^{(1)}\| = 1$ for all k , the hypotheses on the norms of the vectors in Proposition 4.1 are satisfied by simply taking $r_\varphi = r_\psi = 1$, and $\alpha_\varphi = \alpha_\psi = 0$. Then we have $\rho_\varphi = \rho_\psi = \infty$. In other words: $\varphi_1(z)$ and $\psi_1(z)$ coincide and exist for all $z \in \mathbb{C}$.

As for $\varphi_2(z)$ and $\psi_2(z)$ the situation is a bit more complicated, since, $\varphi_{2n+1}^{(2)} = \psi_{2n+1}^{(2)} = 0$, for all $n \in \mathbb{N}_0$. Hence we are in the situation discussed in the Remark at the end of Section 4, with $\mathfrak{K}^c = \{2n, n \geq 0\}$. We find $\tilde{\varphi}_l^{(2)} = \varphi_{2l}^{(2)} = e_l$ and $\tilde{\psi}_l^{(2)} = \psi_{2l}^{(2)} = e_l$, so that $\langle \tilde{\varphi}_l^{(2)}, \tilde{\psi}_n^{(2)} \rangle = \delta_{l,n}$. Then $\tilde{k}_n = 1$ for all n . Also $\tilde{\epsilon}_l = \hat{\epsilon}_{2l} = (2\alpha_1)^{2l} (2l)!$, and

$$\tilde{\varphi}_2(z) = \tilde{\psi}_2(z) = \tilde{N}(|z|) \sum_{l=0}^{\infty} \frac{(z/\alpha_1)^l}{\sqrt{(2l)!}} e_l,$$

which is defined in all of \mathbb{C} . The normalization turns out to be $\tilde{N}(|z|) = \left(\cosh \left(\frac{|z|}{\alpha_1} \right) \right)^{-1/2}$, for all $z \in \mathbb{C}$. The properties of these states (mutual normalization, resolution of the identity, etc.) easily follow from the general results proved in this section.

5 Conclusions

In this paper we have discussed how to use some ideas coming from the theory of intertwining operators and other ideas concerning non self-adjoint Hamiltonians, to construct new exactly solvable models. We have shown that this, in most of the cases, is really different from the standard situation where similarity conditions between different Hamiltonians can be established. We have found that, in all the cases considered here, useful intertwining relations can be deduced. We have also seen that bicoherent states can naturally be introduced, and we have

found conditions for these states to exist and to satisfy several properties which are usually required to mostly all classes of coherent states, like the resolution of the identity and the fact of being eigenstates of certain lowering operators.

We believe that much more can still be deduced within the framework proposed here, in particular for what concerns the bicoherent states or for the deduction of more solvable models. These are two aspects which are presently under deeper investigation.

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- [25] What only exists is a left inverse of X , the matrix $\frac{2}{3}X^\dagger$, but not an inverse, since $X\left(\frac{2}{3}X^\dagger\right) \neq \mathbb{I}$.
- [26] A detailed analysis of similarity of operators in the context of PT-quantum mechanics can be found in [4].
- [27] This might appear strange at a first glance. We refer to Section 2.2 for a simple example in which X^{-1} does not exist, but $(X^\dagger X)^{-1}$ does. In that example, X is an operator between two different, finite-dimensional, Hilbert spaces, \mathbb{C}^3 and \mathbb{C}^2 .
- [28] Incidentally we observe that another intertwining relation does also exist between Θ_1 and Θ_2 . Indeed we have $a\Theta_1 = \Theta_2 a$.